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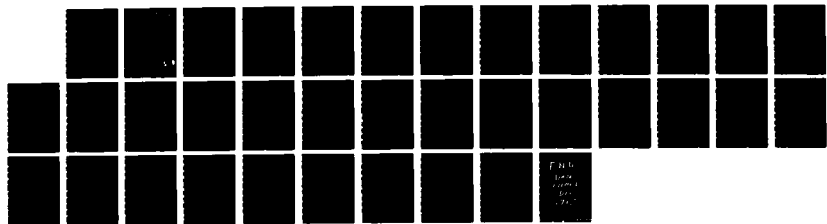
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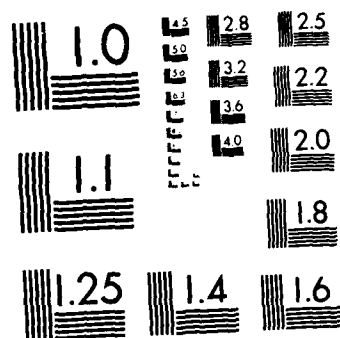
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Almost Periodic Random Sequence in Noise**

by

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Abstract

Under some conditions, the expected numbers of zero-crossings observed in a finite section of a process with a mixed spectrum and in finite sections of its filtered versions, determine the frequencies in the discrete spectrum regardless of the magnitude of the "noise" component.

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# HOC Spectral Analysis of an Almost Periodic Random Sequence in Noise

## 1. Introduction

Consider a stationary random sequence  $\{Z_t\}$ ,  $t = 0, \pm 1, \pm 2, \dots$ , given by the equation

$$Z_t = \sum_{j=1}^P (A_j \cos \omega_j t + B_j \sin \omega_j t) + \xi_t$$

where the amplitudes  $A_j, B_j$ ,  $j = 1, \dots, P$  are random variables and  $\xi_t$  is a random colored noise independent of the  $A_j, B_j$ . The problem addressed in this paper is to determine  $P$  and  $\omega_1, \dots, \omega_P$  from expected zero-crossing counts regardless of the magnitude of the noise term  $\xi_t$ . This can be done under some fairly general conditions by considering sequences of expected zero-crossing counts obtained by repeated filtering of  $\{Z_t\}$ .

Because zero-crossings portray the oscillation in  $\{Z_t\}$  the problem is tantamount to determining  $P$  and the  $\omega$ 's of extremely weak signals buried in noise from the oscillation of the process and its filtered versions. The general message of the present work is that this is indeed possible.

More precisely, we shall be dealing with the so called higher order crossings (HOC). Higher order crossings are zero-crossing counts observed in a process and in its linearly filtered versions. The filtering operation may consist of a single operation applied once or applied repeatedly or it may consist of a succession of different filters. In this regard, repeated differencing and repeated summation play an important role. The shifts or changes in

the spectral distribution resulting from the filtering operation are captured very economically by the higher order crossings, a fact that led to a methodology useful in discrete spectrum analysis as reviewed in Kedem (1986). The present paper is a refinement of the work reported there. Specifically we investigate the convergence of sequences of expected HOC, as well as functions of expected HOC, to the  $\omega$ 's under various conditions on the spectrum of  $\{\xi_t\}$ .

When  $\{Z_t\}$  is Gaussian with mean zero, the oscillation depicted by the expected HOC is equivalent to knowing the correlation function. However, HOC have a somewhat more direct interpretation in terms of the spectrum as can be seen from their spectral representation that will be discussed below.

In the present paper, both  $\{Z_t\}$  and  $\{\xi_t\}$  are assumed to be stationary and Gaussian. The first part of the paper stresses aspects of filter design and direct convergence of HOC to discrete frequencies, while in the second part, consisting of section 4, the convergence is achieved rather indirectly by certain functions of HOC. The main results of this paper are Theorems 3.5 and 4.1 stated in sections 3.3 and 4, respectively. In Theorem 3.5 we assume  $\xi_t$  is white noise,  $p = 1$ , and construct a sequence of expected HOC that converges monotonically to  $\omega_1$  regardless of the signal to noise ratio (that is, the ratio of the standard deviation of the almost periodic component to the standard deviation of the noise). The discussion leading to this result points to the difficulties that arise once noise is added to the almost periodic harmonic component. The basic idea underlying section 4

is the motion of  $A_0$ -intervals. These intervals are instrumental in determining  $p, \omega_1, \dots, \omega_p$  regardless of the type of noise and its magnitude, provided its spectral density is square integrable.

### 1.1. The model and definition of HOC.

Let  $\{X_t\}$  represent the almost periodic harmonic component with  $p$  terms,  $p < \infty$ ,

$$X_t = \sum_{j=1}^p (A_j \cos \omega_j t + B_j \sin \omega_j t).$$

Without loss of generality assume

$$0 < \omega_1 < \omega_2 < \dots < \omega_p < \pi.$$

The  $\{A_j\}, \{B_j\}$  are taken as uncorrelated normal random variables such that

$$\begin{aligned} EA_j &= EB_j = 0, \quad EA_1 A_j = EB_1 B_j = \delta_{1j} \sigma_j^2 \\ EA_i B_j &= 0 \quad \text{for all } i, j. \end{aligned}$$

We assume that  $\{\xi_t\}$  is a stationary zero mean Gaussian process with an absolutely continuous spectrum  $F_\xi$  and spectral density  $f(\omega)$ ,  $-\pi < \omega \leq \pi$ . For each  $t$ ,  $\xi_t$  has a normal distribution with mean 0 and variance  $\sigma_\xi^2$ . It follows that

$$Z_t = X_t + \xi_t$$

is a stationary Gaussian process with a mixed spectrum whose spectral distribution function can be expressed as a sum

$$F(\omega) = F_X(\omega) + F_\xi(\omega), \quad -\pi < \omega \leq \pi,$$

where  $F_X$  is a right continuous step function with jumps of size  $\frac{1}{2} \sigma_j^2$  at  $\pm \omega_j$ ,  $j = 1, \dots, p$ .



Let  $\mathcal{B}$  be the shift operator  $\mathcal{B}Z_t = Z_{t-1}$ . Then repeated differencing can be defined by the operator

$$\nabla^n Z_t = (1-\mathcal{B})^n Z_t = \sum_{k=0}^n \binom{n}{k} (-1)^k Z_{t-k}.$$

Similarly, repeated summation is defined by

$$\nabla^n Z_t = (1+\mathcal{B})^n Z_t = \sum_{k=0}^n \binom{n}{k} Z_{t-k}.$$

Let  $\{Y_t\}$ ,  $t = 0, \pm 1, \dots$ , be any stochastic process and let  $\chi[\cdot]$  be the indicator function. Then the number of zero-crossings in  $Y_1, \dots, Y_N$  is given by

$$D = 2 \sum_{i=1}^N \chi[Y_i \geq 0] - 2 \sum_{i=1}^N \chi[Y_i \geq 0, Y_{i-1} \geq 0] - \chi[Y_1 \geq 0] - \chi[Y_N \geq 0].$$

The number of zero crossings in  $(\nabla^n Z_t)_{t=1}^N$  is denoted by  $D_{n+1}$  and the number of zero crossings in  $(\Delta^n Z_t)_{t=1}^N$  is denoted by  ${}_{n+1}D$ . Then  $D_{n+1}$  and  ${}_{n+1}D$  are examples of HOC. In general, when the linear operation is a filter with transfer function  $H$ , the notation  $D_H$  is used to signify the HOC corresponding to  $H$ .

The problem is to determine  $p$  and  $\omega_1, \dots, \omega_p$  from expected HOC such as  $\{ED_n\}$ ,  $\{E_n D\}$  and  $\{ED_H\}$ .

## 2. Some moment relations

The second order moments of  $\nabla^n Z_t$ ,  $\Delta^n Z_t$  can be expressed quite compactly by introducing the following sequences. Define

$$a_0(n, \theta) = \sum_{j=0}^n \binom{n}{j}^2, \quad b_0(n, \theta) = \sum_{j=0}^n \binom{n}{j} \binom{n}{j+1} \cos(\theta)$$

$$a_k(n, \theta) = 2 \sum_{j=k}^n \binom{n}{j} \binom{n}{j-k} \cos(k\theta), \quad k = 1, \dots, n$$

$$b_k(n, \theta) = \sum_{j=k-1}^n \binom{n}{j} \left[ \binom{n}{j-k+1} \cos((k-1)\theta) + \binom{n}{j-k-1} \cos((k+1)\theta) \right]$$

$k = 1, \dots, n+1.$

Then

$$E(\Delta^n Z_t)^2 = \sum_{k=0}^n a_k(n, 0) R(k)$$

$$E(\Delta^n Z_t \Delta^n Z_{t+1}) = \sum_{k=0}^{n+1} b_k(n, 0) R(k)$$

$$E(\nabla^n Z_t)^2 = \sum_{k=0}^n a_k(n, \pi) R(k)$$

$$E(\nabla^n Z_t \nabla^n Z_{t+1}) = \sum_{k=0}^{n+1} b_k(n, \pi) R(k)$$

where  $R(k) = E Z_t Z_{t+k}$ .

Let  $\rho(k) = R(k)/R(0)$  be the correlation function of  $\{Z_t\}$ . Then from Kedem (1986), the Gaussian assumption implies

$$(2.1) \quad \cos\left(\frac{\pi E D_{n+1}}{N-1}\right) = \frac{\sum_{k=0}^{n+1} b_k(n, \pi) \rho(k)}{\sum_{k=0}^n a_k(n, \pi) \rho(k)},$$

and

$$(2.2) \quad \cos\left(\frac{\pi E (n+1)^D}{N-1}\right) = \frac{\sum_{k=0}^{n+1} b_k(n, 0) \rho(k)}{\sum_{k=0}^n a_k(n, 0) \rho(k)}.$$

Let

$$\mathcal{D}_n = \cos\left(\frac{\pi E D_n}{N-1}\right), \quad n^{\mathcal{D}} = \cos\left(\frac{\pi E_n^{\mathcal{D}}}{N-1}\right).$$

Then  $\rho(0) = 1$ ,  $\rho(1) = \mathcal{D}_1 = {}_1\mathcal{D}$  and so (2.1) and (2.2) imply the existence of a function  $\rho_m(x_1, x_2, \dots, x_m; \theta)$  such that

$$\begin{aligned} \rho(m) &= \rho_m(\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m; \pi) \\ (2.3) \quad &= \rho_m({}_1\mathcal{D}, {}_2\mathcal{D}, \dots, {}_m\mathcal{D}; 0), \quad m = 1, \dots, n \end{aligned}$$

The proof of this fact follows easily by solving (2.1), (2.2) recursively for  $\rho(k)$  starting from  $\rho(0) = 1$ , and noting that

$$b_{n+1}(n, \pi) = (-1)^{n+1}, \quad b_{n+1}(n, 0) = 1.$$

Thus

$$\begin{aligned} \rho(n+1) &= (-1)^{n+1} \sum_{k=0}^n (\mathcal{D}_{n+1} a_k(n, \pi) - b_k(n, \pi)) \rho_k(\mathcal{D}_1, \dots, \mathcal{D}_k; \pi) \\ &= \sum_{k=0}^n ({}_{n+1}\mathcal{D} a_k(n, 0) - b_k(n, 0)) \rho_k({}_1\mathcal{D}, \dots, {}_k\mathcal{D}; 0), \end{aligned}$$

and therefore for each  $n$ , the sequences  $\{\rho(1), \rho(2), \dots, \rho(n)\}$ ,  $\{ED_1, ED_2, \dots, ED_n\}$ , and  $\{E_1^{\mathcal{D}}, E_2^{\mathcal{D}}, \dots, E_n^{\mathcal{D}}\}$  are equivalent.

**Lemma 2.1.** For a zero mean stationary Gaussian process, the sequences  $\{ED_j\}$ ,  $\{E_j^{\mathcal{D}}\}$  and  $\{\rho(n)\}$  are equivalent.

this equivalence relation shows the relevance of HOC in spectral analysis. It should be noted that the correlation function may be obtained from many different HOC sequences, not just those obtained by differencing and summation.

### 3. Filter design and convergence of sequences of expected HOC

We examine the effect of several different filters on zero-crossing counts when the process  $\{Z_t\}$  consists of the harmonic signal only and also when it consists of signal plus white noise.

That is,  $\xi_t = \varepsilon_t$  where  $\{\varepsilon_t\}$  is white noise. In this case

$$f(\omega) = \frac{\sigma^2 \varepsilon}{2\pi}, \quad -\pi < \omega \leq \pi.$$

#### 3.1. A complex filter.

Define a process  $\{Y_t\}$  by

$$Y_t = (1 + e^{i\theta})^n Z_t.$$

Note that  $Y_t$  depends on  $n$  and  $\theta$ . The transfer function and squared gain are given, respectively, by

$$H(\lambda) = (1 + e^{i(\theta - \lambda)})^n$$

$$|H(\lambda)|^2 = 4^n \cos^{2n}\left(\frac{\theta - \lambda}{2}\right).$$

Observe that  $\{Y_t\}$  is complex,

$$Y_t = u_t + i v_t$$

where

$$u_t = \sum_{k=0}^n \binom{n}{k} \cos(k\theta) Z_{t-k}$$

$$v_t = \sum_{k=0}^n \binom{n}{k} \sin(k\theta) Z_{t-k}.$$

It follows that  $E|Y_t|^2$  and  $\text{Re } E Y_t \bar{Y}_{t+1}$  can be written in two equivalent forms as follows. First,

$$\begin{aligned}
E|Y_t|^2 &= \int_{-\pi}^{\pi} |H(\lambda)|^2 dF_X(\lambda) + \int_{-\pi}^{\pi} |H(\lambda)|^2 dF_{\varepsilon}(\lambda) \\
&= \frac{4^n}{2} \sum_{j=1}^p \sigma_j^2 \left[ \cos^{2n} \left( \frac{\theta + \omega_j}{2} \right) + \cos^{2n} \left( \frac{\theta - \omega_j}{2} \right) \right] \frac{4^n \sigma_{\varepsilon}^2}{2\pi} \int_{-\pi}^{\pi} \cos^{2n} \left( \frac{\theta - \lambda}{2} \right) d\lambda
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}(EY_t \bar{Y}_{t+1}) &= \operatorname{Re} \left\{ \int_{-\pi}^{\pi} e^{-i\lambda} |H(\lambda)|^2 dF_X(\lambda) + \int_{-\pi}^{\pi} e^{-i\lambda} |H(\lambda)|^2 dF_{\varepsilon}(\lambda) \right\} \\
&= \frac{4^n}{2} \sum_{j=1}^p \sigma_j^2 \left[ \cos^{2n} \left( \frac{\theta + \omega_j}{2} \right) + \cos^{2n} \left( \frac{\theta - \omega_j}{2} \right) \right] \cos(\omega_j) \\
&\quad + \frac{\sigma_{\varepsilon}^2}{2\pi} 4^n \int_{-\pi}^{\pi} \cos^{2n} \left( \frac{\lambda - \theta}{2} \right) \cos(\lambda) d\lambda.
\end{aligned}$$

Second,

$$\begin{aligned}
E|Y_t|^2 &= Eu_t^2 + Ev_t^2 = \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \cos((k-j)\theta) R(k-j) \\
&= \sum_{k=0}^n a_k(n, \theta) R(k)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}(EY_t \bar{Y}_{t+1}) &= Eu_t u_{t+1} + Ev_t v_{t+1} = \sum_{j=0}^n \sum_{k=0}^n \binom{n}{j} \binom{n}{k} \cos((k-j)\theta) R(k-j+1) \\
&= \sum_{k=0}^{n+1} b_k(n, \theta) R(k).
\end{aligned}$$

From these relations and (2.3) we obtain

$$\begin{aligned}
\frac{\operatorname{Re}(E Y_t \bar{Y}_{t+1})}{E|Y_t|^2} &= \frac{\sum_{k=0}^{n+1} b_k(n, \theta) \rho(k)}{\sum_{k=0}^{n+1} a_k(n, \theta) \rho(k)} \\
&= \frac{\sum_{k=0}^{n+1} b_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; \pi)}{\sum_{k=0}^n a_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; \pi)} \\
&= \frac{\sum_{k=0}^{n+1} b_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; 0)}{\sum_{k=0}^n a_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; 0)} \\
&= \frac{\sum_{j=1}^p \sigma_j^2 \left[ \cos^{2n} \left( \frac{\theta + \omega_j}{2} \right) + \cos^{2n} \left( \frac{\theta - \omega_j}{2} \right) \right] \cos(\omega_j) + \frac{\sigma_\varepsilon^2}{\pi} \int_{-\pi}^{\pi} \cos^{2n} \left( \frac{\lambda - \theta}{2} \right) \cos(\lambda) d\lambda}{\sum_{j=1}^p \sigma_j^2 \left[ \cos^{2n} \left( \frac{\theta + \omega_j}{2} \right) + \cos^{2n} \left( \frac{\theta - \omega_j}{2} \right) \right] + \frac{\sigma_\varepsilon^2}{\pi} \int_{-\pi}^{\pi} \cos^{2n} \left( \frac{\lambda - \theta}{2} \right) d\lambda}
\end{aligned}$$

(3.1)

When  $\{Z_t\}$  is a purely harmonic process, (3.1) provides a way for determining the  $\omega_j$ 's. By choosing an arbitrary  $\theta \in [0, \pi]$ , (3.1) will converge to  $\cos(\omega_r)$  for  $\omega_r$  closest to  $\theta$  as  $n \rightarrow \infty$ .

More precisely we have

**Theorem 3.1.** Assume  $\sigma_\varepsilon^2 = 0$ ,  $\theta \in [0, \pi]$  and suppose

$$|\theta - \omega_r| < \min_{j \neq r} \{|\theta - \omega_j|, 2\pi - \omega_j - \theta\}.$$

Then

$$\frac{\operatorname{Re} E Y_t \bar{Y}_{t+1}}{E|Y_t|^2} \rightarrow \cos(\omega_r).$$

**Proof.** Since  $\cos(x/2)$  is monotone decreasing in  $[0, \pi]$ , the

condition of the theorem implies that

$$\cos\left[\frac{\theta-\omega_r}{2}\right] > |\cos\left[\frac{\theta+\omega_j}{2}\right]|, \quad j \neq r.$$

The assertion now follows from the last expression in (3.1).  $\square$

Since  $\cos(x)$  is monotone in  $[0, \pi]$ , obtaining  $\cos(\omega_r)$  is equivalent to obtaining  $\omega_r$ ,  $r = 1, \dots, p$ . Important special cases occur for  $\theta = 0, \pi$ .

Corollary 3.1. Assume  $\sigma_\varepsilon^2 = 0$ . Then

$$\frac{\pi E D_n}{N-1} \rightarrow \omega_p, \quad n \rightarrow \infty$$

$$\frac{\pi E_n D}{N-1} \rightarrow \omega_1, \quad n \rightarrow \infty.$$

Proof. From (2.1), (2.2) and (3.1), as  $n \rightarrow \infty$ ,

$$\begin{aligned} \mathcal{D}_{n+1} &= \frac{\sum_{k=0}^{n+1} b_k(n, \pi) \rho_k(\mathcal{D}_1, \dots, \mathcal{D}_k; \pi)}{\sum_{k=0}^n a_k(n, \pi) \rho_k(\mathcal{D}_1, \dots, \mathcal{D}_k; \pi)} \rightarrow \cos(\omega_p) \\ n+1 \mathcal{D} &= \frac{\sum_{k=0}^{n+1} b_k(n, 0) \rho_k(\mathcal{D}_1, \dots, \mathcal{D}_k; 0)}{\sum_{k=0}^n a_k(n, 0) \rho_k(\mathcal{D}_1, \dots, \mathcal{D}_k; 0)} \rightarrow \cos(\omega_1), \end{aligned}$$

and note that  $\cos(x)$  is monotone for  $x \in [0, \pi]$ .  $\square$

Unfortunately, the method just outlined breaks down in the presence of noise. To realize the effect of noise define first

$$f_n(\lambda; \theta) = \frac{\cos^{2n}\left[\frac{\theta-\lambda}{2}\right]}{\int_{-\pi}^{\pi} \cos^{2n}\left[\frac{\theta-\omega}{2}\right] d\omega}, \quad -\pi < \lambda \leq \pi.$$

Then we have

**Lemma 3.1.** The sequence of probability densities  $\{f_n(\lambda; \theta)\}$  satisfies

$$\int_{-\pi}^{\pi} f_n(\lambda; \theta) \cos(\lambda) d\lambda \rightarrow \cos(\theta), \quad 0 \leq \theta \leq \pi.$$

**Proof.** Suppose  $0 \leq \theta < \pi$ . Then for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that

$$a = \inf_{|\lambda - \theta| < \delta} \left| \cos\left(\frac{\lambda - \theta}{2}\right) \right| > b = \sup_{|\lambda - \theta| \geq \varepsilon} \left| \cos\left(\frac{\lambda - \theta}{2}\right) \right|.$$

Therefore

$$\int_{|\lambda - \theta| \geq \varepsilon} f_n(\lambda; \theta) d\lambda \leq \frac{2\pi b^{2n}}{2\delta a^{2n}} \rightarrow 0, \quad n \rightarrow \infty.$$

Similarly for  $\theta = \pi$

$$c = \inf_{\pi - \delta < |\lambda| \leq \pi} \left| \cos\left(\frac{\lambda - \pi}{2}\right) \right| > d = \sup_{|\lambda| < \pi - \varepsilon} \left| \cos\left(\frac{\lambda - \pi}{2}\right) \right|$$

so that again

$$\int_{|\lambda| < \pi - \varepsilon} f_n(\lambda; \pi) d\lambda \leq \frac{2\pi d^{2n}}{2\delta c^{2n}} \rightarrow 0, \quad n \rightarrow \infty.$$

Since  $\cos(\lambda)$  is continuous and since  $f_n(\lambda; \theta)$  is for each  $n$  a probability density function with parameter  $\theta$  it follows that

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f_n(\lambda; \theta) \cos(\lambda) d\lambda = \cos(\theta), \quad 0 \leq \theta \leq \pi. \quad \square$$

**Theorem 3.2.** Assume  $\sigma_\varepsilon^2 > 0$ . Then for every  $\theta \in [0, \pi]$



$$\frac{\sum_{k=0}^{n+1} b_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; \pi)}{\sum_{k=0}^n a_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; \pi)} \rightarrow \cos(\theta), \quad n \rightarrow \infty$$

and the same holds if the  $\vartheta_j$  are replaced by the  ${}_j\vartheta$ , and  $\pi$  by 0.

Proof. When  $\omega_j = \theta$  for some  $j$ , the claim follows by bounded convergence from (3.1). Otherwise define

$$b' = \max_{1 \leq j \leq p} \left| \cos \left[ \frac{\theta + \omega_j}{2} \right] \right|$$

and note that the method used in proving Lemma 3.1 yields

$$\frac{(b')^{2n}}{\int_{-\pi}^{\pi} \cos^{2n} \left[ \frac{\theta - \lambda}{2} \right] d\lambda} \rightarrow 0, \quad n \rightarrow \infty.$$

Apply now Lemma 3.1 to the ratio of the two integrals in (3.1).  $\square$

Corollary 3.2. When  $\sigma_\epsilon^2 > 0$ ,

$${}_{n+1}\vartheta \rightarrow 1, \quad \vartheta_{n+1} \rightarrow -1, \quad n \rightarrow \infty,$$

or equivalently

$$\frac{\pi E_{n+1}^D}{N-1} \rightarrow 0, \quad \frac{\pi E_D^{n+1}}{N-1} \rightarrow \pi, \quad n \rightarrow \infty.$$

Proof. The proof follows from (2.1), (2.2).  $\square$

Corollary 3.3.  $\sigma_\epsilon^2 = 0$  if and only if for some  $\theta \in [0, \pi]$ ,  $\theta \notin \{\omega_1, \dots, \omega_p\}$

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n+1} b_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; \pi)}{\sum_{k=0}^n a_k(n, \theta) \rho_k(\vartheta_1, \dots, \vartheta_k; \pi)} \rightarrow \cos(\theta),$$

and the same holds if the  $\vartheta_j$  are replaced by the  $j\vartheta$ , and  $\pi$  by 0.

### 3.2. The Slutsky filter.

For  $k, m, n = 0, 1, 2, \dots$ , put

$$y_t^{(k)} = [(1-\delta)^m (1+\delta)^n]^k z_t, \quad t = 0, 1, \dots$$

Then  $\{y_t^{(k)}\}$  is a real valued stationary Gaussian process with mean 0. The transfer function of this linear operation is given by

$$H_k(\lambda) = (1 - e^{-i\lambda})^{km} (1 + e^{-i\lambda})^{kn}$$

with squared gain

$$|H_k(\lambda)|^2 = 2^{k(m+n)} (1 - \cos \lambda)^{km} (1 + \cos \lambda)^{kn}.$$

The squared gain is symmetric and unimodal in  $[0, \pi]$  with a peak occurring at

$$\lambda_c = \cos^{-1} \left( \frac{n-m}{n+m} \right)$$

Let  $D_{H_{k+1}}$  denote the number of zero-crossings in  $\{y_t^{(k)}\}_{t=1}^N$ , where  $m, n$  are fixed and are chosen in accordance with a prescribed  $\lambda_c$ .

Results similar to those obtained in the previous discussion can be obtained in terms of the HOC  $\{D_{H_k}\}$ ,  $k = 1, 2, \dots$ , the difference being the fact that unlike the case treated earlier the convergence to the  $\omega_j$  can be expressed directly in terms of expected HOC. This is readily seen from the equality

$$\begin{aligned}
\cos\left(\frac{\pi ED_{H_{k+1}}}{N-1}\right) &= \frac{\int_{-\pi}^{\pi} e^{-i\lambda} |H_k(\lambda)|^2 dF(\lambda)}{\int_{-\pi}^{\pi} |H_k(\lambda)|^2 dF(\lambda)} \\
&= \frac{\sum_{j=1}^p \sigma_j^2 [(1-\cos\omega_j)^m (1+\cos\omega_j)^n]^k \cos\omega_j + \frac{\sigma_\varepsilon^2}{\pi} \int_0^\pi [(1-\cos\lambda)^m (1+\cos\lambda)^n]^k \cos\lambda d\lambda}{\sum_{j=1}^p \sigma_j^2 [(1-\cos\omega_j)^m (1+\cos\omega_j)^n]^k + \frac{\sigma_\varepsilon^2}{\pi} \int_0^\pi [(1-\cos\lambda)^m (1+\cos\lambda)^n]^k d\lambda} \\
(3.2)
\end{aligned}$$

Theorem 3.3. Assume  $\sigma_\varepsilon^2 = 0$  and suppose  $m, n$  are such that

$$(1-\cos\omega_r)^m (1+\cos\omega_r)^n > \max_{j \neq r} \{(1-\cos\omega_j)^m (1+\cos\omega_j)^n\}.$$

Then

$$\frac{\pi ED_{H_k}}{N-1} \rightarrow \omega_r, \quad k \rightarrow \infty.$$

Proof. From (3.2)

$$\cos\left(\frac{\pi ED_{H_k}}{N-1}\right) \rightarrow \cos(\omega_r)$$

and  $\cos(x)$  is monotone in  $[0, \pi]$ . □

Corollary 4.1. Assume  $\sigma_\varepsilon^2 = 0$  and suppose  $m, n$  are such that  $\lambda_c$  is sufficiently close to  $\omega_r$ . Then

$$\frac{\pi ED_{H_k}}{N-1} \rightarrow \omega_r, \quad k \rightarrow \infty.$$

Proof. In  $[0, \pi]$   $(1-\cos\lambda)^m (1+\cos\lambda)^n$  is unimodal with the peak occurring at  $\lambda_c$ . □

Thus by varying  $m, n$  so that  $\lambda_c$  lands at or near the  $\omega_j$  we can detect all the frequencies provided the process is purely

harmonic.

Again, as in the previous case, in the presense of noise this procedure breaks down and the expected normalized HOC converge to  $\lambda_c$ .

Theorem 3.4. Assume  $\sigma_\varepsilon^2 > 0$ . Then for  $m, n$  such that  $\lambda_c = \cos^{-1}((n-m)/(n+m))$ , we have

$$\lim_{k \rightarrow \infty} \frac{\pi E D_{H_k}}{N-1} = \lambda_c.$$

Proof. Define a sequence of probability density functions on  $[0, \pi]$ ,

$$\varphi_k(\lambda) = \frac{[(1-\cos \lambda)^m (1+\cos \lambda)^n]^k}{\int_0^\pi [(1-\cos \lambda)^m (1+\cos \lambda)^n]^k d\lambda}, \quad k = 1, \dots$$

Then for every  $\varepsilon > 0$  it is not difficult to see that

$$\int_{|\lambda - \lambda_c| \geq \varepsilon} \varphi_k(\lambda) d\lambda \rightarrow 0, \quad k \rightarrow \infty,$$

from which follows that

$$(3.3) \quad \int_0^\pi \varphi_k(\lambda) \cos(\lambda) d\lambda \rightarrow \cos(\lambda_c) = \frac{n-m}{n+m}.$$

Define now

$$h(\lambda) = (1-\cos \lambda)^m (1+\cos \lambda)^n$$

and without loss of generality assume  $0 < \lambda_c < \pi$ . Consider two cases. First, suppose  $\omega_r = \lambda_c$ . Then obviously  $h(\omega_j) < h(\lambda_c)$ ,  $j \neq r$ , and  $(h(\omega_j)/h(\lambda_c))^k \rightarrow 0$ ,  $k \rightarrow \infty$ . Also, for sufficiently small  $\delta > 0$

$$0 \leq \frac{\int_0^\pi h^k(\lambda) d\lambda}{h^k(\lambda_c)} = \frac{\int_{|\lambda-\lambda_c|>\delta} h^k(\lambda) d\lambda + \int_{|\lambda-\lambda_c|\leq\delta} h^k(\lambda) d\lambda}{h^k(\lambda_c)}$$

$$\leq \frac{\max\{h^k(\lambda_c-\delta), h^k(\lambda_c+\delta)\} \cdot \pi}{h^k(\lambda_c)} + 2\delta \rightarrow 2\delta, \quad k \rightarrow \infty$$

and let  $\delta \rightarrow 0$ . We have shown in fact that

$$\varphi_k(\omega_r) \rightarrow \infty, \quad k \rightarrow \infty.$$

Therefore, from (3.2)  $\cos\left[\frac{\pi ED_{H_{k+1}}}{N-1}\right] \rightarrow \cos \lambda_c, \quad k \rightarrow \infty$ . When  $\omega_j \neq \lambda_c$ , for all  $j$ , then by a similar argument

$$\varphi_k(\omega_j) \rightarrow 0, \quad k \rightarrow \infty$$

so that from (3.2) and (3.3) again  $\cos\left[\frac{\pi ED_{H_{k+1}}}{N-1}\right] \rightarrow \cos(\lambda_c)$ . Thus in general

$$\frac{\pi ED_{H_k}}{N-1} \rightarrow \lambda_c. \quad \square$$

From the last two theorems we have

Corollary 3.4.  $\sigma_\varepsilon^2 \neq 0$  if and only if for  $\lambda_c = \omega_j, \quad j = 1, \dots, p$ ,

$$\frac{\pi ED_{H_k}}{N-1} \rightarrow \lambda_c, \quad k \rightarrow \infty.$$

### 3.3. The alpha filter and the case of a single frequency.

Although the results of the previous two subsections are somewhat pessimistic when  $\sigma_\varepsilon^2 > 0$ , the fast conclusion that the addition of noise makes the detection of  $\omega_1, \dots, \omega_p$  from HOC

impossible is far from being true. Successful detection can be achieved even in the presence of appreciable noise, provided carefully designed filters are used in generating useful HOC sequences. A clue to this effect is furnished by Theorem 3.4. The theorem shows that in the presence of noise, the normalized expected HOC converge to  $\lambda_c$ . The precise reason for this fact is that as  $k \rightarrow \infty$ , more and more spectral weight is given to  $\lambda_c$ , rendering it dominant. As a result the sequence  $\pi ED_{H_k} / (N-1)$ ,  $k = 1, 2, \dots$ , is attracted to  $\lambda_c$  and convergence occurs. When  $\lambda_c$  coincides with an  $\omega_j$  the resulting sequence of expected normalized HOC will converge to it. This shows that by controlling and shifting the spectral mass we can force the sequence of normalized HOC to converge to desired frequencies.

More generally, from (2.1) we obtain the basic spectral representation for the expected number of zero-crossings (since  $\omega_1 > 0$ )

$$(3.4) \quad \cos\left(\frac{\pi ED_1}{N-1}\right) = \frac{\int_0^\pi \cos(\omega) dF(\omega)}{\int_0^\pi dF(\omega)}.$$

(See Kedem (1986) for a discussion and additional references concerning this representation.) From this representation we see that  $\pi ED_1 / (N-1)$  is a weighted average of the spectral support. Therefore  $\pi ED_1 / (N-1)$  will change its location with shifts in the spectral weight  $dF(\omega)$ , and so this quantity can be "directed" to admit values near or at discrete points in the spectral support.

The point of this discussion will now be demonstrated in the special case when  $p = 1$ . We will show that  $\omega_1$  can be detected

by a certain sequence of expected normalized HOC that converge to  $\omega_1$  regardless of the magnitude of the signal to noise ratio.

Let  $p = 1$ ,

$$Z_t = A \cos(\omega_1 t) + B \sin(\omega_1 t) + \varepsilon_t$$

where  $A, B$  are independent  $N(0, \sigma_1^2)$  random variables and independent of the white noise  $\{\varepsilon_t\}$ ,  $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ . Define the  $\alpha$ -filter by

$$Y_t = (1-\alpha)Z_t + \alpha Y_{t-1}, \quad t = 0, \pm 1, \dots,$$

and  $-1 \leq \alpha \leq 1$ . The squared gain of this filter is given by

$$|H_\alpha(\omega)|^2 = \frac{(1-\alpha)^2}{1-2\alpha \cos \omega + \alpha^2}, \quad -\pi < \omega \leq \pi.$$

Let  $D_{H_\alpha}$  be the number of zero-crossings in  $Y_1, \dots, Y_N$ . Fix an  $\alpha_0 \in (-1, 1)$  and let

$$\alpha_{j+1} = \cos \left[ \frac{\pi E D_{H_{\alpha_j}}}{N-1} \right], \quad j = 0, 1, 2, \dots$$

Theorem 3.5. Set  $p = 1$ . Suppose the noise process  $\varepsilon_t = \varepsilon_t$  is white noise. Then as  $j \rightarrow \infty$ ,

$$\alpha_j \rightarrow \cos(\omega_1)$$

or equivalently

$$\frac{\pi E D_{H_{\alpha_j}}}{N-1} \rightarrow \omega_1$$

regardless of the signal to noise ration  $\sigma_1/\sigma_\varepsilon$ .

Proof. Observe that

$$\int_0^{\pi} |H_{\alpha}(\omega)|^2 d\omega = \pi \frac{1-\alpha}{1+\alpha}$$

and

$$\int_0^{\pi} \cos(\omega) |H_{\alpha}(\omega)|^2 d\omega = \alpha \pi \frac{1-\alpha}{1+\alpha}.$$

Therefore, from (3.4)

$$\begin{aligned} \cos \left[ \frac{\pi E D_{H_{\alpha}}}{N-1} \right] &= \frac{\int_0^{\pi} \cos(\omega) |H_{\alpha}(\omega)|^2 dF(\omega)}{\int_0^{\pi} |H_{\alpha}(\omega)|^2 dF(\omega)} \\ &= \frac{\int_0^{\pi} \cos(\omega) |H_{\alpha}(\omega)|^2 dF_X(\omega) + \int_0^{\pi} \cos(\omega) |H_{\alpha}(\omega)|^2 dF_{\epsilon}(\omega)}{\int_0^{\pi} |H_{\alpha}(\omega)|^2 dF_X(\omega) + \int_0^{\pi} |H_{\alpha}(\omega)|^2 dF_{\epsilon}(\omega)} \\ (3.5) \quad &= \frac{\frac{(1-\alpha)\sigma_1^2}{1-2\alpha \cos \omega_1 + \alpha^2} \cos(\omega_1) + \frac{\sigma_{\epsilon}^2}{1+\alpha} \alpha}{\frac{(1-\alpha)\sigma_1^2}{1-2\alpha \cos \omega_1 + \alpha^2} + \frac{\sigma_{\epsilon}^2}{1+\alpha}} \end{aligned}$$

and we see that  $\cos \left[ \frac{\pi E D_{H_{\alpha}}}{N-1} \right]$  is a weighted average of  $\cos(\omega_1)$

and  $\alpha$ . Suppose  $\alpha_0 \leq \cos(\omega_1)$ . Then

$$\alpha_0 \leq \cos \left[ \frac{\pi D_{H_{\alpha_0}}}{N-1} \right] \leq \cos(\omega_1)$$

and

$$\cos \left[ \frac{\pi E D_{H_{\alpha_0}}}{N-1} \right] \leq \cos \left[ \frac{\pi E D_{H_{\alpha_1}}}{N-1} \right] \leq \cos(\omega_1)$$



or more generally

$$\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_j \leq \dots \leq \cos(\omega_1).$$

Thus  $\{\alpha_j\}$  is a monotone increasing and bounded sequence which converges to  $\alpha^*$  say. But then from (3.5) we have

$$(3.6) \quad \alpha^* - \cos(\omega_1) = \frac{\frac{\sigma_\epsilon^2}{1+\alpha^*} (\alpha^* - \cos(\omega_1))}{\frac{(1-\alpha^*)\sigma_1^2}{1 - 2\alpha^* \cos(\omega_1) + \alpha^{*2}} + \frac{\sigma_\epsilon^2}{1+\alpha^*}}.$$

Suppose  $\alpha^* - \cos(\omega_1) \neq 0$ . then dividing both sides of (3.6) by  $\alpha^* - \cos(\omega_1)$  leads to

$$1 = \frac{\frac{\sigma_\epsilon^2}{1+\alpha^*}}{\frac{(1-\alpha^*)\sigma_1^2}{1 - 2\alpha^* \cos \omega_1 + \alpha^{*2}} + \frac{\sigma_\epsilon^2}{1+\alpha^*}} < 1$$

and hence to a contradiction. Therefore

$$\alpha^* = \cos(\omega_1)$$

or

$$\frac{\pi \text{ED}_{H_{\alpha^*}}}{N-1} = \omega_1.$$

When  $\alpha_0 > \cos(\omega_1)$ , the sequence  $\{\alpha_j\}$  is monotone decreasing and bounded and by the same argument converges to  $\cos(\omega_1)$  from above.

From the proof of this theorem we can see that any filter with transfer function  $H_\theta(\omega)$  which depends on a parameter  $\theta \in (-1, 1)$  can be used in the detection of  $\omega_1$ . All that is needed is that

$$\int_0^\pi \cos(\omega) |H_\theta(\omega)|^2 d\omega = \theta \int_0^\pi |H_\theta(\omega)|^2 d\omega.$$

When  $p = 2$  we can prove in the same way that for large  $j$ ,  $\pi \text{ED}_{H_{\alpha_j}} / (N-1)$  admits values between  $\omega_1$  and  $\omega_2$ .

Theorem 3.6. Let  $p = 2$  and choose an  $\alpha_0 \in (-1, 1)$ . Then regardless of the signal to noise ratio  $(\sigma_1^2 + \sigma_2^2)^{1/2} / \sigma_\epsilon$ ,

$$\alpha_j \rightarrow \alpha \in [\cos \omega_2, \cos \omega_1]$$

or equivalently

$$\frac{\pi \text{ED}_{H_{\alpha_j}}}{N-1} \rightarrow \omega \in [\omega_1, \omega_2].$$

Proof. As before  $\alpha_j$  is monotone and bounded and thus converges to  $\alpha$ , where

$$\cos \omega_2 \leq \alpha = \frac{\sigma_1^2 (1 - 2\alpha \cos \omega_2 + \alpha^2) \cos \omega_1 + \sigma_2^2 (1 - 2\alpha \cos \omega_1 + \alpha^2) \cos \omega_2}{\sigma_1^2 (1 - 2\alpha \cos \omega_2 + \alpha^2) + \sigma_2^2 (1 - 2\alpha \cos \omega_1 + \alpha^2)} \leq \cos \omega_1. \quad \square$$

### 3.4. Detection of periodicities by HOC.

The generalization of Theorem 3.5 to the case  $p \geq 2$  requires more sophisticated filtering which we shall not pursue here. In practice, the generalization takes a somewhat different route. Since the normalized HOC tend to admit values near or at dominant frequencies, the central idea is to evaluate the periodogram at the normalized HOC. The combination of HOC and the periodogram in this manner has been reviewed and discussed in Kedem (1986).

#### 4. A complete solution

Theorem 3.5 shows that in one special case it is possible to determine a single frequency from HOC regardless of the signal to noise ratio. In this section we give a general solution to the problem of determining  $p, \omega_1, \dots, \omega_p$  in the presence of any colored Gaussian noise  $\{Z_t\}$  with continuous density  $f(\omega)$ . But rather than using HOC directly we use functions of HOC. Generally speaking, it is sometimes more beneficial to use functions of HOC and in particular the correlations  $\{\rho(k)\}$  which by (2.3) are functions of expected HOC. Thus, we will show that the oscillation in  $\{Z_t\}$  as depicted by the expected HOC, obtained by repeated differencing, determines the discrete frequencies and their number.

Recall (2.3) and define

$$\begin{aligned} h_N(\lambda) &= h_N(\lambda; \mathcal{D}_1, \dots, \mathcal{D}_{N-1}) \\ &= N^{-3/4} \left[ 1 + 2 \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \rho_n(\mathcal{D}_1, \dots, \mathcal{D}_n; \pi) \cos(n\lambda) \right] \\ &= N^{-3/4} \left[ 1 + 2 \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \rho_n(\mathcal{D}_1, \dots, \mathcal{D}_n; 0) \cos(n\lambda) \right]. \end{aligned}$$

We shall investigate the asymptotic behavior of  $\{h_N(\lambda)\}$  and show that this sequence of functions of expected HOC determines  $p$  and  $\omega_1, \dots, \omega_p$  regardless of the signal to noise ratio where the noise is any colored noise. This is done by showing that as  $N \rightarrow \infty$ , the sum represented by  $h_N(\lambda)$  vanishes for  $\lambda \neq \omega_j, j = 1, \dots, p$ , but diverges otherwise. In this regard the key idea is the definition of an  $A_0$ -interval.

Definition 4.1. Let  $\{g_N(\lambda)\}$  be a sequence of continuous func-

tions on  $[0, \pi]$ . Let  $A_0$  be a positive number and  $I_N(\alpha, \beta)$  a subinterval of  $[0, \pi]$ . We say that  $I_N = (\alpha, \beta)$  is an  $A_0$ -interval of  $g_N(\lambda)$  if the following definitions are satisfied.

- (a)  $g_N(\lambda) < A_0, \lambda \in I_N$
- (b)  $g_N(\alpha) = g_N(\beta) = A_0$
- (c) The Lebesgue measure of  $I_N > N^{-1/2}$ .

We will show that asymptotically, the number of  $A_0$ -intervals of  $h_N(\lambda)$  is equal to  $p-1$ .

We start off with a lemma due to Wang (1983). The upper bound given here is an improvement over the one given there.

**Lemma 4.1.** Let  $\{\xi(n), n = 0, \pm 1, \dots\}$  be a real valued stationary Gaussian process with mean zero, and square integrable spectral density function  $f(\lambda)$ . Then for any  $\alpha \geq 0$  and  $k$  such that  $k = 2\alpha + 3/2$  we have the inequality

$$E\left\{\sup_{\lambda} \frac{1}{N^k} \left| \sum_{n=1}^N \xi(n) n^{\alpha} e^{in\lambda} \right|^2\right\} \leq (4\pi + \sqrt{4\pi}) \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda}.$$

Proof. For every fixed  $\lambda$

$$\begin{aligned} \left| \sum_{n=1}^N \xi(n) n^{\alpha} e^{in\lambda} \right|^2 &= \sum_{m=1}^N \xi(m) m^{2\alpha} + 2 \sum_{m=1}^{N-1} \xi(m) \xi(m+1) m^{\alpha} (m+1)^{\alpha} \cos \lambda + \\ &\quad \dots + 2 \xi(N) \xi(1) N^{\alpha} \cos(N-1)\lambda \\ &\leq 2 \sum_{n=0}^{N-1} \sum_{m=1}^{N-n} \xi(m) \xi(m+n) m^{\alpha} (m+n)^{\alpha} \cos(n\lambda) \\ &\leq 2 \sum_{n=0}^{N-1} \left| \sum_{m=1}^{N-n} \xi(m) \xi(m+n) m^{\alpha} (m+n)^{\alpha} \right| \end{aligned}$$

$$\begin{aligned}
& \leq 2 \sum_{n=0}^{N-1} \left| \sum_{m=1}^{N-n} [\xi(m)\xi(m+n) - R(n)] m^\alpha (n+m)^\alpha \right| + \\
(4.1) \quad & + 2 \sum_{n=0}^{N-1} \left| \sum_{m=1}^{N-n} R(n) m^\alpha (m+n)^\alpha \right|
\end{aligned}$$

where  $R(n) \equiv E\xi(m)\xi(m+n)$ . Note that the last two expressions in the sequence of inequalities are independent of  $\lambda$ . The first of these terms can be simplified by introducing the lag-process

$$Y_n(m) = \xi(m)\xi(m+n) - R(n), \quad m = 0, \pm 1, \dots,$$

where  $EY_n(m) = 0$  and by the Gaussian assumption

$$\begin{aligned}
EY_n(m)Y_n(m+k) &= E\xi(m)\xi(m+n)\xi(m+k)\xi(m+k+n) - R^2(n) \\
&= R^2(n) + R^2(k) + R(n-k)R(n+k) - R^2(n) \\
&= R^2(k) = R(n-k)R(n+k)
\end{aligned}$$

so that  $\{Y_n(m)\}$  for each fixed lag  $n$  is wide-sense stationary with mean zero. Also, if we define

$$R_{Y_n}(k) = EY_n(m)Y_n(m+k), \quad k = 0, \pm 1, \dots,$$

then by Cauchy-Schwarz inequality and orthogonality of sines and cosines

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |R_{Y_n}(k)| &\leq \sum_{k=-\infty}^{\infty} (R^2(k) + R^2(k)) \\
&= 2 \sum_{k=-\infty}^{\infty} R^2(k) \leq 4\pi \int_{-\pi}^{\pi} f^2(\lambda) d\lambda.
\end{aligned}$$

Therefore  $\{Y_n(m)\}$  has a continuous spectral density  $f_n(\lambda)$ ,

say, such that

$$(4.2) \quad \sup_{\lambda} f_n(\lambda) \leq 4\pi \int_{-\pi}^{\pi} f^2(\omega) d\omega.$$

uniformly in  $n$ . Going back to the first of the last two expressions in (4.1) and by invoking in succession first Cauchy-Schwarz inequality and then (4.2) we have

$$\begin{aligned} E \left| \sum_{m=1}^{N-n} [\xi(m)\xi(m+n) - R(n)] m^{\alpha}(m+n)^{\alpha} \right| &= E \left| \sum_{m=1}^{N-n} Y_n(m) m^{\alpha}(m+n)^{\alpha} \right| \leq \left\{ E \left| \sum_{m=1}^{N-n} Y_n(m) m^{\alpha}(m+n)^{\alpha} \right|^2 \right\}^{1/2} \\ &= \left\{ \int_{-\pi}^{\pi} \left| \sum_{m=1}^{N-n} m^{\alpha}(m+n)^{\alpha} e^{im\lambda} \right|^2 f_n(\lambda) d\lambda \right\}^{1/2} \\ &\leq \left\{ 4\pi \int_{-\pi}^{\pi} f^2(\lambda) d\lambda \int_{-\pi}^{\pi} \left| \sum_{m=1}^{N-n} m^{\alpha}(m+n)^{\alpha} e^{im\lambda} \right|^2 d\lambda \right\}^{1/2} \\ &\leq \left\{ 4\pi \int_{-\pi}^{\pi} f^2(\lambda) d\lambda \cdot 2\pi \cdot N \cdot N^{2\alpha} \cdot N^{2\alpha} \right\}^{1/2} \\ &= 2\pi \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \cdot N^{2\alpha+1/2}. \end{aligned}$$

For the last expression in (4.1) we have

$$\begin{aligned} \sum_{n=0}^{N-1} \left| \sum_{m=1}^{N-n} R(n) m^{\alpha}(m+n)^{\alpha} \right| &\leq \sum_{n=0}^{N-1} |R(n)| \sum_{m=1}^{N-n} m^{\alpha}(m+n)^{\alpha} \\ &\leq N^{1/2} \left\{ \sum_{k=-\infty}^{\infty} R^2(n) \right\}^{1/2} N \cdot N^{2\alpha} \leq \sqrt{2\pi \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \cdot N^{2\alpha+3/2}. \end{aligned}$$

Returning to (4.1) equipped with these new expressions which are independent of  $\lambda$ , we see that

$$\begin{aligned} & E \left\{ \sup_{\lambda} \left| \sum_{n=1}^N \zeta(n) n^{\alpha} e^{in\lambda} \right|^2 \right\} \\ & \leq 2N \cdot 2\pi \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \cdot N^{2\alpha+1/2} + 2 \sqrt{2\pi \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \cdot N^{2\alpha+3/2} \\ & = (4\pi + \sqrt{4\pi}) \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \cdot N^{2\alpha+3/2}. \quad \square \end{aligned}$$

Next we show that  $h_N(\lambda)$  tends to be inflated as  $N \rightarrow \infty$  in small neighborhoods of the  $\omega_j$  and vanishes elsewhere.

Define the set  $A$  by

$$A = \bigcap_{j=1}^p \{ \lambda \in [0, \pi] : N^{-3/4} \leq |\lambda - \omega_j| \leq 2\pi - N^{-3/4} \}.$$

This set is made of  $[0, \pi]$  minus neighborhoods of size  $2N^{-3/4}$  around the  $\omega_j$ .

Theorem 4.1. Let  $R_Z(k) = E Z_t Z_{t+k}$ . Then

(a) for each fixed  $j$

$$\sup_{|\lambda - \omega_j| \leq \frac{\pi}{N}} h_N(\lambda) > \frac{2\sigma_j^2 N^{1/4}}{\pi^2 R_Z(0)}.$$

(b) For  $N$  sufficiently large such that

1.  $[\frac{1}{2}N^{-3/4} / \sin(\frac{1}{2}N^{-3/4})] \leq \frac{1}{2\sqrt{5}}$
2. for  $\omega_1 > 0$ ,  $N^{-3/4} \leq \omega_1$
3. for  $\omega_p < \pi$ ,  $N^{-3/4} \leq \pi - \omega_p$

the inequality

$$\sup_{\lambda \in \mathcal{A}} h_N(\lambda) \leq \frac{1}{R_Z(0)} \left[ 5 \sum_{j=1}^p \sigma_j^2 + (4\pi + \sqrt{4\pi}) \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \right] N^{-1/4}$$

holds.

Proof.

$$\begin{aligned} R_Z(0) h_N(\lambda) &= N^{-3/4} \left[ 1 + 2 \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \rho_n(\mathcal{D}_1, \dots, \mathcal{D}_n; \pi) \cos(n\lambda) \right] R_Z(0) \\ &= N^{-3/4} \left[ R_Z(0) + 2 \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) R_Z(n) \cos(n\lambda) \right] \\ &= N^{-7/4} \left[ \sum_{m=1}^N R_Z(0) + 2 \sum_{n=1}^{N-1} (N-n) R_Z(n) \cos(n\lambda) \right] \\ &= N^{-7/4} E \left[ \sum_{m=1}^N Z_n e^{in\lambda} \right] \left[ \sum_{m=1}^N Z_n e^{-in\lambda} \right] \\ &= N^{-7/4} E \left| \sum_{m=1}^N Z_n e^{in\lambda} \right|^2. \end{aligned}$$

Define

$$\eta(\lambda) = \left| \sum_{m=1}^N Z_n e^{in\lambda} \right|^2 = \frac{\sin^2\left(\frac{N}{2}\lambda\right)}{\sin^2\left(\frac{1}{2}\lambda\right)}.$$

Then

$$\begin{aligned} E \left| \sum_{n=1}^N Z_n e^{-in\lambda} \right|^2 &= \int_{-\pi}^{\pi} \left| \sum_{n=1}^N e^{in\omega} e^{-in\lambda} \right|^2 dF_X(\omega) + \int_{-\pi}^{\pi} \left| \sum_{n=1}^N e^{in(\omega-\lambda)} \right|^2 f(\omega) d\omega \\ &= \frac{1}{2} \sum_{j=1}^p \sigma_j^2 \left\{ \left| \sum_{n=1}^N e^{in(\omega_j-\lambda)} \right|^2 + \left| \sum_{n=1}^N e^{in(\omega_j+\lambda)} \right|^2 \right\} + E \left| \sum_{n=1}^N \xi(n) e^{-in\lambda} \right|^2 \end{aligned}$$



$$= \frac{1}{2} \sum_{j=1}^p \sigma_j^2 \left\{ \frac{\sin^2 \frac{N}{2}(\omega_j - \lambda)}{\sin^2 \frac{1}{2}(\omega_j - \lambda)} + \frac{\sin^2 \frac{N}{2}(\omega_j + \lambda)}{\sin^2 \frac{1}{2}(\omega_j + \lambda)} \right\} + E \left| \sum_{n=1}^N z(n) e^{-in\lambda} \right|^2.$$

Now observe that since  $\eta(\lambda)$  has a maximum at  $\lambda = 0$ ,

$$\inf_{\lambda \in (0, \pi/n)} \eta(\lambda) = \eta(\pi/N) = \frac{1}{\sin^2 \frac{\pi}{2N}} > \frac{4N^2}{\pi^2}.$$

Therefore, over the  $\lambda$ -interval  $|\lambda - \omega_j| < \pi/N$

$$\begin{aligned} h_N(\lambda) &= \frac{N^{-7/4}}{R_Z(0)} E \left| \sum_{n=1}^N z_n e^{-in\lambda} \right|^2 \\ &> \frac{N^{-7/4}}{R_Z(0)} \frac{1}{2} \sigma_j^2 \eta(\lambda - \omega_j) \\ &> \frac{N^{-7/4}}{R_Z(0)} \frac{1}{2} \sigma_j^2 \frac{4N^2}{\pi^2} \end{aligned}$$

or, for a fixed  $j$

$$\inf_{|\lambda - \omega_j| < \frac{\pi}{N}} h_N(\lambda) > \frac{2\sigma_j^2 N^{1/4}}{\pi^2 R_Z(0)}.$$

To prove the second part of the theorem observe that for  $\lambda \in A$  and  $N$  that satisfies 1., 2., 3. we have

$$\frac{1}{2}N^{-3/4} \leq \frac{1}{2}|\lambda \pm \omega_j| \leq \pi - \frac{1}{2}N^{-3/4}.$$

By using Lemma 4.1 with  $\alpha = 0$  we finally obtain for  $\lambda \in A$

$$h_N(\lambda) = \frac{N^{-7/4}}{R_Z(0)} E \left| \sum_{n=1}^N z_n e^{-in\lambda} \right|^2$$

$$\begin{aligned}
& \leq \frac{N^{-7/4}}{R_Z(0)} \frac{1}{2} \sum_{j=1}^p \sigma_j^2 \left[ \frac{1}{\sin^2 \frac{1}{2}(\omega_j - \lambda)} + \frac{1}{\sin^2 \frac{1}{2}(\omega_j + \lambda)} \right] \\
& \quad + \frac{N^{-7/4}}{R_Z(0)} E \left| \sum_{n=1}^N \xi(n) e^{-in\lambda} \right|^2 \\
& \leq \frac{N^{-7/4}}{R_Z(0)} \frac{1}{2} \sum_{j=1}^p \sigma_j^2 \left[ \frac{1}{\sin^2(\frac{1}{2} N^{-3/4})} + \frac{1}{\sin^2(\frac{1}{2} N^{-3/4})} \right] \\
& \quad + \frac{N^{-7/4}}{R_Z(0)} N^{-7/4} (4\pi + \sqrt{4\pi}) \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \cdot N^{3/2} \\
& \leq \frac{N^{-7/4}}{R_Z(0)} \frac{1}{2} \cdot 2 \sum_{j=1}^p \sigma_j^2 5N^{3/2} + \frac{N^{-1/4}}{R_Z(0)} (4\pi + \sqrt{4\pi}) \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \\
& = \frac{N^{-1/4}}{R_Z(0)} \left[ 5 \sum_{j=1}^p \sigma_j^2 + (4\pi + \sqrt{4\pi}) \sqrt{2 \int_{-\pi}^{\pi} f^2(\lambda) d\lambda} \right]
\end{aligned}$$

and note that the last expression is independent of  $\lambda \in A$ .  $\square$

With the help of this theorem we can finally determine  $p$  and the  $\omega$ 's by constructing the  $A_0$ -intervals of  $h_N(\lambda)$  and letting  $N$  increase. It is easy to see that for any fixed  $N$ , the number of  $A_0$ -intervals of  $h_N(\lambda)$  is finite and that the  $A_0$ -intervals are well defined. This is the subject of the next result.

First we obtain  $p$ .

**Theorem 4.2.** Let  $A_0$  be an arbitrarily chosen positive constant and define

$$p_N = 1 + \text{the number of } A_0\text{-intervals of } h_N(\lambda).$$

Then regardless of the signal to noise ratio

$$\lim_{N \rightarrow \infty} \hat{p}_N = p.$$

Proof. By Theorem 4.1, when  $N$  is sufficiently large and for  $\lambda$  such that

$$\lambda \in [\omega_1 + N^{-3/4}, \omega_{i+1} - N^{-3/4}], \quad 1 \leq i \leq p-1$$

we have

$$h_N(\lambda) < A_0,$$

and

$$(\omega_{i+1} - \omega_1) - 2N^{-3/4} \geq \min_{1 \leq j \leq p-1} (\omega_{j+1} - \omega_j) - 2N^{-3/4} > N^{-1/2}.$$

On the other hand for large  $N$

$$\lim_{|\lambda - \omega_j| < \frac{\pi}{N}} h_N(\lambda) > A_0.$$

Also, note that the intervals where  $h_N(\lambda)$  must cross  $A_0$  at least once are such that

$$0 < (\omega_1 + N^{-3/4}) - (\omega_1 + \frac{\pi}{N}) = N^{-3/4}(1 - \pi N^{-1/4}) < N^{-1/2}$$

and

$$0 < (\omega_1 - \pi/N) - (\omega_1 - N^{-3/4}) = N^{-3/4}(1 - \pi N^{-1/4}) < N^{-1/2}.$$

It follows that between  $\omega_1$  and  $\omega_{i+1}$ ,  $i = 1, \dots, p-1$ , there is exactly one  $A_0$ -interval of  $h_N(\lambda)$ , provided  $N$  is large enough. Now consider the two extreme points  $\omega_1, \omega_p$ . Assume that  $0 < \omega_1 < \omega_p < \pi$ . Then for large  $N$

$$\sup_{0 \leq \lambda < \omega_1 - N^{-3/4}} h_N(\lambda) < A_0$$

and

$$\sup_{\omega_p + N^{-3/4} < \lambda \leq \pi} h_N(\lambda) < A_0.$$

Therefore,  $h_N(\lambda)$  does not cross level  $A_0$  over  $[0, \omega_1 - N^{-3/4}) \cup (\omega_p + N^{-3/4} < \lambda \leq \pi]$  and consequently  $h_N(\lambda)$  has no  $A_0$ -intervals over  $[0, \omega_1)$  and over  $(\omega_p, \pi]$ . Thus  $\hat{p}_N \rightarrow p, N \rightarrow \infty$ .  $\square$

From the proof of Theorem 4.2. it is clear that the  $A_0$ -intervals of  $h_N(\lambda)$

$$I_{j,N} = (b_{j,N}, a_{j+1,N}), \quad j = 1, 2, \dots, \hat{p}_N - 1$$

satisfy for large  $N$  and for some  $a_{1,N}, b_{p,N}, 0 \leq a_{1,N} \leq \omega_1 < b_{1,N} < a_{2,N} < \omega_2 < b_{2,N} < \dots < b_{p-1,N} < a_{p,N} < \omega_p \leq b_{p,N} \leq \pi$  where

$$\max_{1 \leq j \leq p} (b_{j,N} - a_{j,N}) < 2N^{-3/4}.$$

We finally have

Theorem 4.3. Suppose  $\{a_{j,N}\}, \{b_{j,N}\}$  correspond to the  $A_0$ -intervals of  $h_N(\lambda)$ . Let

$$\hat{\omega}_j \in [a_{j,N}, b_{j,N}].$$

Then for  $\beta < 3/4$

$$\lim_{N \rightarrow \infty} N^\beta |\hat{\omega}_j - \omega_j| = 0$$

regardless of the signal to noise ratio.

The last three results show that

Corollary 4.1. In the Gaussian case,  $p, \omega_1, \dots, \omega_p$  are completely determined by  $\{ED_j\}_{j=1}^\infty$  provided the spectral density of the noise term is square integrable.

### References

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